Types of Triangles

A triangle is a closed figure made of three line segments. Triangles are classified according to angles and sides. See the chart below.

<table>
<thead>
<tr>
<th>Angles</th>
<th>Acute Triangle</th>
<th>Right Triangle</th>
<th>Obtuse Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angles</td>
<td>All angles are acute</td>
<td>One right angle</td>
<td>One obtuse angle</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sides</th>
<th>Equilateral Triangle</th>
<th>Isosceles Triangle</th>
<th>Scalene Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sides</td>
<td>All sides equal</td>
<td>Two sides equal</td>
<td>No sides equal</td>
</tr>
</tbody>
</table>

Two triangles are **similar** if they have exactly the same shape but not necessarily the same size. The corresponding angles are the same. The sides are proportional.

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

Triangles that are both same size and the same shape are called congruent triangle. All congruent triangles are similar but two similar triangles are not necessarily congruent.
Properties of Triangles

An important property of triangles deals with the sum of the measures of the angles within the triangle.

**The sum of the measures of the angles of any triangle is 180°.**

\[ \alpha + \beta + 90° = 180° \]
\[ \alpha + \beta = 90° \]
\[ m\angle A + m\angle B + m\angle C = 180° \]

All triangles can be decomposed into right triangles. The right triangle is the foundation of trigonometry. We will restrict our study in this course to right triangles. Consider the right triangle shown below. When labeling a right triangle, the vertices are labeled with capital letters which denote the angles. In particular, C is always the label for the right angle. The sides are labeled with lower case letters. The longest side is always opposite the right angle. This side is called the **hypotenuse** and is labeled c. The other two sides, called legs, are labeled according the angle that sits across from it.

Since the sum of the angles in a triangle is 180°, then angle A and angle B are complementary angles.

\[ A + B + C = 180° \text{ but } C = 90° \]
\[ A + B + 90° = 180° \]
\[ A + B = 90° \]

The sides are related by the Pythagorean Theorem:

**Pythagorean Theorem:** \[ a^2 + b^2 = c^2 \]

**Example 1** Complete the triangle:
Trigonometric Functions

There are six trigonometric functions that relate the angle of a right triangle to a real number. The real number is the ratio of two sides of the right triangle. There are six ratios, hence six trigonometric functions. Referring to the triangle to the right the six ratios are:

\[
\frac{a}{c}, \frac{b}{c}, \frac{c}{a}, \frac{b}{b}, \frac{a}{b}, \frac{b}{a}
\]

We will define the trigonometric functions for an acute angle \(\theta\). The relative positions of the sides of the triangle to angle \(\theta\) are important. One side is opposite \(\theta\), one side is adjacent to \(\theta\) and the third side is the hypotenuse. See the diagram below. Note: the hypotenuse is always the longest side and is always opposite the 90° angle.

**Right Triangle Definitions of Trigonometric Functions**

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Name</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine of theta</td>
<td>(\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}})</td>
<td>Cosecant of theta</td>
<td>(\csc \theta = \frac{\text{Hypotenuse}}{\text{Opposite}})</td>
</tr>
<tr>
<td>Cosine of theta</td>
<td>(\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}})</td>
<td>Secant of theta</td>
<td>(\sec \theta = \frac{\text{Hypotenuse}}{\text{Adjacent}})</td>
</tr>
<tr>
<td>Tangent of theta</td>
<td>(\tan \theta = \frac{\text{Opposite}}{\text{Adjacent}})</td>
<td>Cotangent of theta</td>
<td>(\cot \theta = \frac{\text{Adjacent}}{\text{Opposite}})</td>
</tr>
</tbody>
</table>

**Example 2** Find the values of the six trigonometric functions of angle theta.

\[
\sin \theta = \frac{5}{13} \quad \csc \theta = \frac{13}{5} \quad \tan \theta = \frac{5}{12}
\]

\[
\cos \theta = \frac{12}{13} \quad \sec \theta = \frac{13}{12} \quad \cot \theta = \frac{12}{5}
\]
Two preliminary observations will help you to understand and memorize the definitions:

1. There are three pairs of reciprocals in the definitions: cosine and secant; sine and cosecant; and tangent and cotangent.
2. An expression such as \( \cos \theta \) really means \( \cos(\theta) \). Where \( \cos \) or cosine is the mane of the function and \( \theta \) is an input.

**Example 3** Given \( AB = 13; \ AC = 5 \) and \( BC = 12 \).
Does this represent a right triangle? (verify)

Find each of the following:

\[
\begin{align*}
\sin A &= \ldots \quad \sin B &= \ldots \\
\cos A &= \ldots \quad \cos B &= \ldots \\
\tan A &= \ldots \quad \tan B &= \ldots \\
\csc A &= \ldots \quad \csc B &= \ldots \\
\sec A &= \ldots \quad \sec B &= \ldots 
\end{align*}
\]

**Example 4** Find \( \cos \theta \):

\[
\begin{align*}
\cos \theta &= \ldots
\end{align*}
\]
**Example 5** Find the values of the other five trigonometric functions of \( \theta \), given \( \sin \theta = \frac{2}{3} \).

\[
\begin{align*}
\cos \theta &= \ldots & \sec \theta &= \ldots \\
\tan \theta &= \ldots & \cot \theta &= \ldots \\
\csc \theta &= \ldots
\end{align*}
\]

**Special Angles**

There are angles for which trigonometric values can be computed using a right triangle. Consider the angle whose measure is 45° or \( \frac{\pi}{4} \). In a right triangle, if one angle is 45°, so must the other. Therefore the right triangle will be isosceles. Let one leg of the triangle measure 1 unit. The other leg must also be 1 unit. By the Pythagorean Theorem:

\[
c^2 = a^2 + b^2 \\
c^2 = 1^2 + 1^2 \\
c^2 = 2 \\
c = \sqrt{2}
\]

We now have a triangle that can be used to find the exact values of the six trigonometric functions of 45° or \( \frac{\pi}{4} \).

**Example 6** Use the triangle above and the definition of the trigonometric functions to complete the following:

\[
\begin{align*}
\sin 45° &= \ldots & \csc 45° &= \ldots \\
\cos 45° &= \ldots & \sec 45° &= \ldots \\
\tan 45° &= \ldots & \cot 45° &= \ldots
\end{align*}
\]
Now consider an equilateral triangle. An equilateral triangle is also equiangular. This implies that each angle has a measure of $60^\circ$ or $\frac{\pi}{3}$. This, however is not a right triangle. To construct a right triangle, drop an altitude from one of the vertices to the opposite side. This will bisect the triangle as well as the $60^\circ$ angle. You now have two right triangles whose angles are $30^\circ$, $60^\circ$ and $90^\circ$ or $\frac{\pi}{6}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. See the diagram below.

Let the length of the sides of the original equilateral triangle be 2 units. The altitude bisects the opposite side so one leg of the right triangle will be 1 unit. Apply the Pythagorean theorem to compute the length of the second leg.

\[ x^2 + 1^2 = 2^2 \]
\[ x^2 + 1 = 4 \]
\[ x^2 = 3 \]
\[ x = \sqrt{3} \]
This triangle allows us to evaluate the six trigonometric functions for two angles: 30° and 60°.

Example 7 Use the triangle above and the definition of the trigonometric functions to complete the following:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>sin 30°=</td>
<td>sin 60°=</td>
</tr>
<tr>
<td>cos 30°=</td>
<td>cos 60°=</td>
</tr>
<tr>
<td>tan 30°=</td>
<td>tan 60°=</td>
</tr>
<tr>
<td>csc 30°=</td>
<td>csc 60°=</td>
</tr>
<tr>
<td>sec 30°=</td>
<td>sec 60°=</td>
</tr>
<tr>
<td>cot 30°=</td>
<td>cot 60°=</td>
</tr>
</tbody>
</table>

These angles: \(30° = \frac{\pi}{6}\), \(60° = \frac{\pi}{3}\) and \(45° = \frac{\pi}{4}\) are called special angles and will be used frequently. You should learn to construct either right triangle and properly label the sides to use as a reference. With sufficient practice you will memorize the values computed above.

Trigonometric Relationships

There are many relationships among the trigonometric functions. Such relationships are called identities. An identity is an equation that is true for all relevant values of the variable. For example \(2x(x - 1) = 2x^2 - 2x\) is true for all values of \(x\).
Reciprocal Identities

You should have observed earlier that each trigonometric function has a reciprocal. For example \( \sin \theta \) is the reciprocal of \( \csc \theta \) and vice versa. Cosine of theta and secant of theta are reciprocals of each other; tangent and cotangent are reciprocals. These reciprocal relationships can be expressed more mathematically as follows:

<table>
<thead>
<tr>
<th>Reciprocal Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \csc \theta = \frac{1}{\sin \theta} )</td>
</tr>
<tr>
<td>( \sec \theta = \frac{1}{\cos \theta} )</td>
</tr>
<tr>
<td>( \cot \theta = \frac{1}{\tan \theta} )</td>
</tr>
</tbody>
</table>

Cofunction Identities

Another relationship is the cofunction identity. Recall that the two acute angles in any right triangle are complementary angles. So if one acute angle is labeled \( \theta \), the other angle must be \( 90^\circ - \theta \). Notice the following:

1. 60 and 30 are complementary angles
2. \( \sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2} \)

More generally:

- \( \sin \theta = \frac{a}{c} \) and \( \cos(90^\circ - \theta) = \frac{a}{c} \) are equal.

- \( \tan \theta = \frac{a}{b} \) and \( \cot(90^\circ - \theta) = \frac{a}{b} \) are equal.

- \( \sec \theta = \frac{c}{b} \) and \( \csc(90^\circ - \theta) = \frac{c}{b} \) are equal.

If two angles are complementary, then the sine of (either) one equal the cosine of the other.
This is the cofunction relation. “Co” is short for *complementary*. The pairs of cofunctions are designated by the name:

- Sine ⇔ **Cosine**
- Tangent ⇔ **Cotangent**
- Secant ⇔ **Cosecant**

### Cofunction Identities

- \( \sin(90^\circ - \theta) = \cos \theta \iff \cos(90^\circ - \theta) = \sin \theta \)
- \( \tan(90^\circ - \theta) = \cot \theta \iff \cot(90^\circ - \theta) = \tan \theta \)
- \( \sec(90^\circ - \theta) = \csc \theta \iff \csc(90^\circ - \theta) = \sec \theta \)

### Example 8

Refer to the right triangle on the right to find each of the following:

- \( \cos \theta = \) 
- \( \sin\left(\frac{\pi}{2} - \theta\right) = \) 
- \( \tan(90^\circ - \theta) = \) 
- \( \csc(90^\circ - \theta) = \)

### Quotient Identities

The third set of identities are called the quotient identities. They are:

- \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) and \( \cot \theta = \frac{\cos \theta}{\sin \theta} \). Refer to the right triangle to verify the first. The second is easily established by the reciprocal relation.

- \( \frac{\sin \theta}{\cos \theta} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \) where \( \frac{a}{b} = \tan \theta \)
### Quotient Identities

- \( \tan \theta = \frac{\sin \theta}{\cos \theta} \)
- \( \cot \theta = \frac{\cos \theta}{\sin \theta} \)

### Pythagorean Identities

The fourth set of identities are derived from the Pythagorean Theorem. Hence they are referred to as the Pythagorean Identities. The most important one to memorize is \( \sin^2 \theta + \cos^2 \theta = 1 \).

Note: \( \sin^2 \theta = (\sin \theta)^2 \) or \( \sin \theta \cdot \sin \theta \)

This can be derived directly from the definitions of the trig functions. We will start with the left hand side: \( \sin^2 \theta + \cos^2 \theta \). Refer the triangle shown to the right. \( \sin \theta = \frac{b}{c} \); \( \cos \theta = \frac{a}{c} \). Substituting we have:

\[
\sin^2 \theta + \cos^2 \theta = \left( \frac{b}{c} \right)^2 + \left( \frac{a}{c} \right)^2 = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{b^2 + a^2}{c^2} = \frac{c^2}{c^2} = 1
\]

There are two variations of this equation that can be derived using properties of equality. You will need to recognize these variations as well. They are: \( \sin^2 \theta = 1 - \cos^2 \theta \) and \( \cos^2 \theta = 1 - \sin^2 \theta \).

The other two remaining Pythagorean identities can be obtained by dividing the original equation \( \sin^2 \theta + \cos^2 \theta = 1 \) by either \( \sin \theta \) or \( \cos \theta \) and applying previously established identities.

**Dividing through by \( \cos^2 \theta \):**

\[
\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} \\
\tan^2 \theta + 1 = \sec^2 \theta
\]

**Dividing through by \( \sin^2 \theta \):**

\[
\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} \\
1 + \sec^2 \theta = \csc^2 \theta
\]
The Calculator

The first thing you need to know about is the **mode**. Put the calculator in degree mode when you are working in degrees. Put the calculator in radian mode when working with radians.

The calculator has three buttons to represent the trigonometric functions. They are: \[ \sin \quad \cos \quad \tan \]

To evaluate a sine, cosine or tangent of an angle in degree mode, press the function key then type in the angle. For example:

\[ \sin 30 \]

You should have gotten 0.5. This is a good way to check your mode.

In order to calculate secant, cosecant, and cotangent, you will need to use the reciprocal identities and the reciprocal button on the calculator. The reciprocal button will be one of the following: \[ x^{-1} \quad \frac{1}{x} \]

To evaluate \( \csc 30^\circ \) we need to apply: \( \csc 30^\circ = \frac{1}{\sin 30^\circ} \). The keystrokes will be:

\[ \sin 30 \quad x^{-1} \]

**Example** Evaluate each of the following using a calculator. Round to 4 decimal places.

\[
\begin{align*}
\tan 17^\circ &= \quad \sec 25^\circ = \\
cot 29^\circ &= \quad \sin^2 70^\circ = \\
\cos \frac{\pi}{5} &= \quad \frac{2 \tan 10^\circ}{\tan 15^\circ} = \\
\sin 3 + \cos 1 &= \quad \frac{3 \cos 20^\circ}{2 \cos 70^\circ + \cos 20^\circ} = \\
\sin^2 \pi + \cos^2 \pi &= \\
\end{align*}
\]
1. Complete the triangle and evaluate each of the trigonometric functions.

\[ a = \quad \]

\[
\begin{align*}
\text{a. } \sin \theta &= \quad \\
\text{b. } \cos \theta &= \quad \\
\text{c. } \tan \theta &= \quad \\
\text{d. } \cot \theta &= \quad \\
\text{e. } \sec \theta &= \quad \\
\text{f. } \csc \theta &= \quad \\
\text{g. } \sin \beta &= \quad \\
\text{h. } \cos \beta &= \quad \\
\text{i. } \tan \beta &= \quad \\
\text{j. } \cot \beta &= \quad \\
\text{k. } \sec \beta &= \quad \\
\text{l. } \csc \beta &= \quad
\end{align*}
\]
2. If AB = 3 and AC = 1, compute the six trigonometric functions of angle A.

a. \( \sin A = \) \\
b. \( \cos A = \) \\
c. \( \tan A = \) \\
d. \( \cot A = \) \\
e. \( \sec A = \) \\
f. \( \csc A = \)

3. Use a calculator to evaluate each of the following. Round to three decimal places.

a. \( \cos 65^\circ = \) \\
b. \( \tan 15^\circ = \) \\
c. \( \csc 40^\circ = \) \\
d. \( \sec 18.5^\circ = \) \\
e. \( \sin 3^\circ = \) \\
f. \( \cot 1^\circ = \) \\
g. \( \csc \pi = \) \\
h. \( \sec \frac{\pi}{2} = \) \\
i. \( \tan 2 = \) \\
j. \( \cot 1.24 = \)
4. Find the EXACT values of each of the following trigonometric functions. Do not use a calculator

   a. \( \sin 60\degree + \cos 30\degree = \) __________

   b. \( \sin 30\degree - \csc^2 45\degree = \) __________

   c. \( \sin 45\degree + \cos 45\degree = \) __________

   d. \( \cos^2 60\degree + \sin^2 \frac{\pi}{3} = \) __________

   e. \( \tan^2 60\degree - \sec^2 60\degree = \) __________

   f. \( \csc \frac{\pi}{3} + \cot \frac{\pi}{4} = \) __________

5. Use the given information to determine the values of the remaining five trigonometric functions.

   a. \( \sin \theta = \frac{3}{4} \)

      \( \cos \theta = \) __________

      \( \tan \theta = \) __________

      \( \cot \theta = \) __________

      \( \sec \theta = \) __________

      \( \csc \theta = \) __________
b. \( \cos \beta = \frac{\sqrt{3}}{5} \)

\[ \sin \beta = \boxed{\sqrt{1 - \cos^2 \beta}} \]

\[ \tan \beta = \frac{\sin \beta}{\cos \beta} \]

\[ \sec \beta = \frac{1}{\cos \beta} \]

\[ \csc \beta = \frac{1}{\sin \beta} \]

\[ \cot \beta = \frac{1}{\tan \beta} \]
Applications of Trigonometric Functions

To solve applied trigonometry problems, follow the same procedure as solving a right triangle.

<table>
<thead>
<tr>
<th>Solving an Applied Trigonometry Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1:</strong> Draw a sketch and label it with the given information. Label the quantity to be found with a variable.</td>
</tr>
<tr>
<td><strong>Step 2:</strong> Use the sketch to write an equation relating the given quantities to the variables.</td>
</tr>
<tr>
<td><strong>Step 3:</strong> Solve the equation and check that your answer makes sense.</td>
</tr>
</tbody>
</table>

**Example 1**

If the altitude of an isosceles triangle is 25cm and each of the two equal angles measures $17^\circ$, how long are the two equal sides?

**Example 2**

To measure the width of a river without crossing, a surveyor stands on one bank and sights straight across to a tree on the opposite bank. The surveyor then paces 100 yards along the bank and sights the same tree, measuring an angle of $35^\circ$ from the bank to his line of sight. How wide is the river?
Example 3

A wire stretches from the top of a vertical pole to a point on a ground 16 feet from the base of the pole. If the wire makes an angle of 62° with the ground, determine the height of the pole and the length of the wire.

Many applications of right triangles involve angles of elevation or depression.

Definitions:

**Angle of elevation:** the angle measured from the horizontal upward to the line of site.

**Angle of depression:** the angle measured from the horizontal downward to the line of site.

Be careful when interpreting the angle of depression. Both the angle of elevation and the angle of depression are measured between the line of sight and a *horizontal* line.

Example 4

A helicopter hovers 800 feet above a building. The pilot spots another building in the distance. If the angle of depression is 35°, what is the distance between the buildings?
**Example 5**

From a point on the floor the angle of elevation to the top of a door is 47°, while the angle of elevation to the ceiling above the door is 59°. If the ceiling is 10ft above the floor, what is the vertical dimension of the door?

**Example 6**

A surveyor determines that the angle of elevation to the top of a building is 40°. Moving 100 feet further away from the building the surveyor determines the angle of elevation to be 35°. Determine the height of the building.
1. A ladder, which is leaning against the side of a building, forms an angle of 50° with the ground. If the top of the ladder is 14 feet up the side of the building, a) how long is the ladder? b) How far from the base of the building is the foot of the ladder?

2. A lighthouse L stands 3 miles from the nearest point P on the shore. Point Q is located down the shoreline and PQ is perpendicular to PL. Determine the distance from P to a point Q along the shore if \( \angle PQL = 35° \). How far from the lighthouse is Q?

3. A ranger spots a fire from a 73-foot tower in Yellowstone National Park. She measures the angle of depression to be 10°. How far is the fire from the tower?
4. In a lighthouse at a point 130 ft. above the surface of the water, an observer measures the angle of depression down to a buoy to be 6°. Find the distance from the base of the lighthouse to the buoy.

5. From the top of a 100 ft. Building a man observes a car moving toward him. If the angle of depression of the car changes from 15° to 33° during the period of observation, how far does the car travel?

6. A surveyor determines that the angle of elevation of a mountain peak is 35°. Moving 1000 feet further away from the mountain the surveyor determines the angle of elevation to be 30°. Determine the height of the mountain.